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A stability theory for second-order nonsmooth dynamical systems with application to friction problems

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Abstract

A LaSalle's Invariance Theory for a class of first-order evolution variational inequalities is developed. Using this approach, stability and asymptotic properties of important classes of second-order dynamic systems are studied. The theoretical results of the paper are supported by examples in nonsmooth Mechanics and some numerical simulations.

Résumé

Dans cet article, la théorie d'invariance de LaSalle est généralisée pour une classe d'inéquations variationnelles d'évolution du premier ordre. Des résultats de stabilité (au sens de Lyapunov) et d'attractivité sont ensuite obtenus pour des systèmes dynamiques du second ordre non réguliers. Une extension du théorème de Lagrange aux systèmes conservatifs non réguliers est également proposée. Enfin, quelques exemples et simulations numériques illustrent les principaux résultats théoriques.

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1. Introduction

The stability of stationary solutions of unilateral dynamic systems constitutes a very important topic in Mathematics and Engineering which has recently attracted important research interest (see, e.g., [1–6,9,11,14,15]).

The aim of this paper is to provide a mathematical theory applicable to the study of dynamic systems of the form

$$M\frac{\mathrm{d}^2q}{\mathrm{d}t^2}(t)+C\frac{\mathrm{d}q}{\mathrm{d}t}(t)+Kq(t)\in -H_1\partial\Phi\left(H_2\frac{\mathrm{d}q}{\mathrm{d}t}(t)\right),\quad \text{a.e. } t\geqslant t_0, \tag{1}$$

where $t_0\in\mathbb{R}$ is fixed, $\Phi:\mathbb{R}^l\to\mathbb{R}\cup\{+\infty\}$ is a proper convex and lower semicontinuous function and $M, C, K\in\mathbb{R}^{m\times m}$, $H_1\in\mathbb{R}^{m\times l}$, $H_2\in\mathbb{R}^{l\times m}$ are given matrices. Here $\partial\Phi$ denotes the convex subdifferential of Φ.

The model in (1) plays an important role in Unilateral Mechanics. Indeed, the motion of various systems (with m degrees of freedom) having frictional contact can be written in the compact form (1). Then M is the mass matrix of the system, C is the viscous damping matrix and K is the stiffness matrix. The vector q is a vector in which the generalized coordinates are listed. Generally the matrices M, C and K are symmetric and positive semidefinite matrices. In many cases M and K can be positive definite and $H_2=H_1^{\mathrm{T}}$.

The term $H_1\partial\Phi(H_2.)$ has been introduced in order to model the unilaterality of the contact induced by friction forces. Indeed, it is now well known that contact with friction can be described by a relation of the form:

$$f\in -H_1\partial\Phi\left(H_2\frac{\mathrm{d}q}{\mathrm{d}t}\right), \tag{2}$$

where f denotes the vector of friction forces.

Indeed, friction force which opposes motion, is a complicated combination of all the force components that are distributed along the mechanical links like flat surfaces, bearings, etc. Friction characteristics can also be influenced by lubrication, temperature, a possible gear mechanical system, etc. It has been observed that experimental friction characteristics versus velocities approximated by making use of spline polynomial functions may include vertical segments. If a "graph" $(\frac{\mathrm{d}q_i}{\mathrm{d}t}, -f_i)$ is monotone then it can usually be recovered by a subdifferential relation of the form $f_i\in-\partial\varphi_i(\frac{\mathrm{d}q_i}{\mathrm{d}t})$ where φ_i is a convex function. This is, for example, the case of the famous Coulomb model (see, e.g., [5,7,8,11,13]). Most discrete systems are made of point masses connected to each others and a whole formulation of the friction dynamic leads usually to a mathematical model like the one given in (2).

In this paper, we give also some results applicable to the model:

$$M\frac{\mathrm{d}^2q}{\mathrm{d}t^2}(t)+\Pi'\big(q(t)\big)\in -H_1\partial\Phi\left(H_1^{\mathrm{T}}\frac{\mathrm{d}q}{\mathrm{d}t}(t)\right),\quad \text{a.e. } t\geqslant t_0, \tag{3}$$

where $\Pi\in C^1(\mathbb{R}^m;\mathbb{R})$ and $M\in\mathbb{R}^{m\times m}$ is assumed symmetric and positive definite.

The model in (3) concerns mechanical systems involving conservative forces $Q = -\Pi'(q)$ where Π is the potential energy of the system.

It seems that the literature does not yet propose a general mathematical approach to study the stability of problems (1) and (3). This is the aim of this work.

In this paper, we give conditions on the data M, K, C, H_1, H_2 and Φ so as to ensure the existence and uniqueness of a solution $q(\cdot; t_0, q_0, \dot{q}_0)$ of (1) satisfying given initial conditions $q(t_0) = q_0$ and $\frac{dq}{dt}(t_0) = \dot{q}_0$. Then we give conditions ensuring that any stationary solution of (1) is stable (in the sense of Lyapunov). Finally, we discuss some asymptotic properties of the model. More precisely, we give conditions ensuring that

$$\lim_{\tau \to +\infty} \mathrm{d}\big(q(\tau; t_0, q_0, \dot{q}_0), \mathcal{W}\big) = 0 \quad \text{and} \quad \lim_{\tau \to +\infty} \frac{\mathrm{d}q}{\mathrm{d}t}(\tau; t_0, q_0, \dot{q}_0) = 0,$$

where $\mathcal{W} := \{\bar{q} \in \mathbb{R}^m\colon\ K\bar{q} \in -H_1 \partial\Phi(0)\}$ denotes the set of stationary solutions of (1).

To prove such results, we first give conditions ensuring that problem (1) can be reduced to a first-order evolution variational inequality. Next we develop a theory extending LaSalle invariance principle (see, e.g., [16]) to first-order evolution variational inequalities.

Sections 2 and 3 concern this class of first-order dynamic systems. In Section 2, we recall a stability theorem which has been recently proved in [6]. In Section 3, we prove some general invariance theorem applicable to a large class of first-order evolution variational inequalities. In Section 4, we use the results of Sections 2 and 3 so as to discuss the stability of the system in (1).

The results of Sections 2 and 3 are also used in Section 5 so as to prove a theorem extending the famous Lagrange's theorem (see, e.g., [12]) to the model in (3).

Finally, some illustrative small-sized examples in Mechanics are presented in Section 6.

2. First-order dynamic systems

In this section, we deal with the following general class of first-order dynamic systems.

Let $\varphi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper convex and lower semicontinuous function. The notations $D(\varphi)$ and $D(\partial\varphi)$ stand for the domain of φ and the domain of the subdifferential $\partial\varphi$ of φ, respectively, i.e.,

$$D(\varphi) := \big\{x \in \mathbb{R}^n\colon\ \varphi(x) < +\infty\big\}$$

and

$$D(\partial\varphi) := \big\{x \in \mathbb{R}^n\colon\ \partial\varphi(x) \neq \emptyset\big\}.$$

Recall that

$$D(\partial\varphi) \subset D(\varphi), \qquad \overline{D(\partial\varphi)} = \overline{D(\varphi)}.$$

Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous operator such that for some $\omega \geqslant 0$, $F + \omega I$ is monotone. Here I denotes the identity mapping on $\mathbb{R}^n$. For $(t_0, x_0) \in \mathbb{R} \times D(\partial\varphi)$, we consider the problem $P(t_0, x_0)$: Find a function $t \mapsto x(t)$ $(t \geqslant t_0)$ with $x \in C^0([t_0, +\infty); \mathbb{R}^n)$, $\frac{\mathrm{d}x}{\mathrm{d}t} \in L^\infty_{\mathrm{loc}}(t_0, +\infty; \mathbb{R}^n)$ and such that

$$\begin{cases} x(t) \in D(\partial\varphi), & t \geqslant t_0, \\ \left\langle \dfrac{\mathrm{d}x}{\mathrm{d}t}(t) + F\big(x(t)\big), v - x(t) \right\rangle + \varphi(v) - \varphi\big(x(t)\big) \geqslant 0, & \forall v \in \mathbb{R}^n, \text{ a.e. } t \geqslant t_0, \\ x(t_0) = x_0. \end{cases} \tag{4}$$

Here $\langle \cdot, \cdot \rangle$ denotes the Euclidean scalar product in $\mathbb{R}^n$. The corresponding norm is denoted by $\| \cdot \|$. It follows from standard convex analysis that (4) can be rewritten equivalently as the differential inclusion:

$$\frac{\mathrm{d}x}{\mathrm{d}t}(t) + F\big(x(t)\big) \in -\partial\varphi\big(x(t)\big). \tag{5}$$

Remark 1. Note that if $F : \mathbb{R}^n \to \mathbb{R}^n$ is Lipschitz continuous with Lipschitz constant $k > 0$ then F is continuous and $F + kI$ is monotone.

Let us first specify some conditions ensuring the existence and uniqueness of the initial value problem $P(t_0, x_0)$. The following existence and uniqueness result is essentially a consequence of Kato's theorem [10]. We refer the reader to [6, Corollary 2.2] for the details.

Theorem 1. *Let $\varphi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper convex and lower semicontinuous function and let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous operator such that for some $\omega \geqslant 0$, $F + \omega I$ is monotone. Let $t_0 \in \mathbb{R}$ and $x_0 \in D(\partial\varphi)$ be given. Then there exists a unique $x \in C^0([t_0, +\infty); \mathbb{R}^n)$ such that*

$$\frac{\mathrm{d}x}{\mathrm{d}t} \in L^\infty_{\mathrm{loc}}\big(t_0, +\infty; \mathbb{R}^n\big), \tag{6}$$

$$x \text{ is right-differentiable on } [t_0, +\infty), \tag{7}$$

$$x(t_0) = x_0, \tag{8}$$

$$x(t) \in D(\partial\varphi), \quad t \geqslant t_0, \tag{9}$$

$$\left\langle \frac{\mathrm{d}x}{\mathrm{d}t}(t) + F\big(x(t)\big), v - x(t) \right\rangle + \varphi(v) - \varphi\big(x(t)\big) \geqslant 0, \quad \forall v \in \mathbb{R}^n, \text{ a.e. } t \geqslant t_0. \tag{10}$$

Suppose that the assumptions of Theorem 1 are satisfied and denote by $x(\cdot; t_0, x_0)$ the unique solution of problem $P(t_0, x_0)$. We prove below that for $t \geqslant t_0$ fixed, the application $x(t; t_0, \cdot)$ is uniformly continuous on $D(\partial\varphi)$. This property will be used later in Section 3. Let us first recall some Gronwall inequality that is used in our next result (see, e.g., Lemma 4.1 in [17]).

Lemma 1. *Let $T > 0$ be given and let $a, b \in L^1(t_0, t_0 + T; \mathbb{R})$ with $b(t) \geqslant 0$ a.e. $t \in [t_0, t_0 + T]$. Let the absolutely continuous function $w : [t_0, t_0 + T] \to \mathbb{R}_+$ satisfy:*

$$(1 - \alpha)w'(t) \leqslant a(t)w(t) + b(t)w^\alpha(t), \quad \text{a.e. } t \in [t_0, t_0 + T],$$

where $0 \leqslant \alpha < 1$. Then

$$w^{1-\alpha}(t) \leqslant w^{1-\alpha}(t_0)\mathrm{e}^{\int_{t_0}^t a(\tau)\,\mathrm{d}\tau} + \int_{t_0}^{t} \mathrm{e}^{\int_s^t a(\tau)\,\mathrm{d}\tau} b(s)\,\mathrm{d}s, \quad \forall t \in [t_0, t_0 + T].$$

Theorem 2. *Let the assumptions of Theorem 1. Let $\tau \geqslant t_0$ be fixed. The application*

$$x(\tau; t_0, \cdot) : x_0 \mapsto x(\tau; t_0, x_0)$$

is uniformly continuous on $D(\partial\varphi)$.

Proof. Let $\tau \geqslant t_0$ be fixed. Let $\varepsilon > 0$ be given and set,

$$\delta := \frac{\varepsilon}{\sqrt{\mathrm{e}^{2\omega(\tau - t_0)}}}.$$

We claim that if $x_0, x_0^\star \in D(\partial\varphi)$, $\|x_0 - x_0^\star\| \leqslant \delta$ then $\|x(\tau; t_0, x_0) - x(\tau; t_0, x_0^\star)\| \leqslant \varepsilon$. Indeed, let us set $x(t) := x(t; t_0, x_0)$ and $x^\star(t) := x(t; t_0, x_0^\star)$. We know that

$$\left\langle \frac{\mathrm{d}x}{\mathrm{d}t}(t) + F\big(x(t)\big), v - x(t) \right\rangle + \varphi(v) - \varphi\big(x(t)\big) \geqslant 0, \quad \forall v \in \mathbb{R}^n, \text{ a.e. } t \geqslant t_0 \tag{11}$$

and

$$\left\langle \frac{\mathrm{d}x^\star}{\mathrm{d}t}(t) + F\big(x^\star(t)\big), z - x^\star(t) \right\rangle + \varphi(z) - \varphi\big(x^\star(t)\big) \geqslant 0, \quad \forall z \in \mathbb{R}^n, \text{ a.e. } t \geqslant t_0. \tag{12}$$

Setting $v = x^\star(t)$ in (11) and $z = x(t)$ in (12), we obtain the relations:

$$-\left\langle \frac{\mathrm{d}x}{\mathrm{d}t}(t) + F\big(x(t)\big), x^\star(t) - x(t) \right\rangle + \varphi\big(x(t)\big) - \varphi\big(x^\star(t)\big) \leqslant 0, \quad \text{a.e. } t \geqslant t_0 \tag{13}$$

and

$$\left\langle \frac{\mathrm{d}x^\star}{\mathrm{d}t}(t) + F\big(x^\star(t)\big), x^\star(t) - x(t) \right\rangle + \varphi\big(x^\star(t)\big) - \varphi\big(x(t)\big) \leqslant 0, \quad \text{a.e. } t \geqslant t_0. \tag{14}$$

It results that

$$\begin{aligned}\left\langle \frac{\mathrm{d}(x^\star - x)}{\mathrm{d}t}(t), x^\star(t) - x(t) \right\rangle &\leqslant \langle \omega x^\star(t) - \omega x(t), x^\star(t) - x(t)\rangle \\ &\quad - \langle [F+\omega I]\big(x^\star(t)\big) - [F+\omega I]\big(x(t)\big), x^\star(t) - x(t)\rangle \\ &\leqslant \omega \big\| x^\star(t) - x(t)\big\|^2, \quad \text{a.e. } t \geqslant t_0.\end{aligned}$$

Recalling that $x \in C^0([t_0, +\infty); \mathbb{R}^n)$ and $\frac{\mathrm{d}x}{\mathrm{d}t} \in L^\infty_{\mathrm{loc}}(t_0, +\infty; \mathbb{R}^n)$, we may write:

$$\frac{\mathrm{d}}{\mathrm{d}t}\big\|x^\star(t) - x(t)\big\|^2 \leqslant 2\omega \big\|x^\star(t) - x(t)\big\|^2, \quad \text{a.e. } t \geqslant t_0. \tag{15}$$

We may apply Lemma 1 with $T > \tau - t_0$, $\alpha = 0$, $b(\cdot) = 0$, $a(\cdot) = 2\omega$ and $w(\cdot) = \|x^\star(\cdot) - x(\cdot)\|^2$ to get:

$$\big\|x^\star(t) - x(t)\big\|^2 \leqslant \big\|x_0^\star - x_0\big\|^2 \mathrm{e}^{2\omega(t-t_0)}, \quad \forall t \in [t_0, t_0 + T]. \tag{16}$$

It follows that

$$\big\|x^\star(\tau) - x(\tau)\big\| \leqslant \delta\sqrt{\mathrm{e}^{2\omega(\tau - t_0)}} = \varepsilon. \qquad \square$$

Suppose now in addition to the assumptions of Theorem 1 that

$$0 \in D(\partial\varphi), \quad F(0) \in -\partial\varphi(0). \tag{17}$$

Then

$$x(t; t_0, 0) = 0, \quad \forall t \geqslant t_0,$$

i.e., the trivial stationary solution 0 is the unique solution of problem $P(t_0, 0)$.

We may now define as in [6] the stability of the trivial solution. The stationary solution 0 is called stable if small perturbations of the initial condition $x(t_0) = 0$ lead to solutions which remain in the neighborhood of 0 for all $t \geqslant t_0$, precisely:

Definition 1. The equilibrium point $x = 0$ is said to be stable in the sense of Lyapunov if, for every $\varepsilon > 0$, there exists $\eta = \eta(\varepsilon) > 0$ such that for any $x_0 \in D(\partial\varphi)$ with $\|x_0\| \leqslant \eta$ the solution $x(\cdot; t_0, x_0)$ of problem $P(t_0, x_0)$ satisfies $\|x(t; t_0, x_0)\| \leqslant \varepsilon$, $\forall t \geqslant t_0$.

If in addition the trajectories of the perturbed solutions are attracted by 0 then we say that the stationary solution is asymptotically stable, precisely:

Definition 2. The equilibrium point $x = 0$ is asymptotically stable if (1) it is stable and (2) there exists $\delta > 0$ such that for any $x_0 \in D(\partial\varphi)$ with $\|x_0\| \leqslant \delta$ the solution $x(\cdot; t_0, x_0)$ of problem $P(t_0, x_0)$ fulfills

$$\lim_{t \to +\infty} \big\|x(t; t_0, x_0)\big\| = 0.$$

Note that the equilibrium point $x = 0$ is said *attractive* (respectively *globally attractive*) as soon as part (2) of Definition 2 holds (respectively for any $x_0 \in D(\partial\varphi)$).

Let us now recall a general abstract theorem of stability in terms of generalized Lyapunov functions $V \in C^1(\mathbb{R}^n;\mathbb{R})$. The following result is a particular case of the one proved in [6]. Here, for $r > 0$, we denote by B_r the closed ball of radius r, i.e., $B_r := \{x \in \mathbb{R}^n\colon \|x\| \leqslant r\}$.

Theorem 3. *Let the assumptions of Theorem* 1 *together with condition* (17). *Suppose that there exists* $\sigma > 0$ *and* $V \in C^1(\mathbb{R}^n;\mathbb{R})$ *such that*

(1) $V(x) \geqslant a(\|x\|),\ x \in D(\partial\varphi) \cap B_\sigma$, *with* $a : [0,\sigma] \to \mathbb{R}$ *satisfying* $a(t) > 0, \forall t \in (0,\sigma)$;
(2) $V(0) = 0$;
(3) $\langle F(x), V'(x)\rangle + \varphi(x) - \varphi(x - V'(x)) \geqslant 0,\ x \in D(\partial\varphi) \cap B_\sigma$.

Then the trivial solution of (9)–(10) *is stable.*

Various applications of Theorem 3 can be found in [6,9]. For example, engineering systems described by a SPR transfer function and a feedback branch containing a sector static nonlinearity are discussed in [9].

We end this section by remarking that some of the hypothesis stated in Theorem 3 can also be used to obtain some additional information on the set of stationary solutions of (9)–(10).

Let us here denote by $\mathcal{S}(F,\varphi)$ the set of stationary solutions of (9)–(10), that is:

$$\mathcal{S}(F,\varphi) := \bigl\{z \in D(\partial\varphi)\colon \bigl\langle F(z), v - z\bigr\rangle + \varphi(v) - \varphi(z) \geqslant 0, \forall v \in \mathbb{R}^n\bigr\}.$$

Condition (17) ensures that $0 \in \mathcal{S}(F,\varphi)$. Let $V \in C^1(\mathbb{R}^n;\mathbb{R})$ be given. We set:

$$E(F,\varphi,V) := \bigl\{x \in D(\partial\varphi)\colon \bigl\langle F(x), V'(x)\bigr\rangle + \varphi(x) - \varphi\bigl(x - V'(x)\bigr) = 0\bigr\}. \tag{18}$$

Let us end this section by showing that condition (3) in Theorem 3 has some consequences on the qualitative properties of the stationary solutions of (9)–(10).

Proposition 1. *Let the assumptions of Theorem* 1 *together with condition* (17). *Let* Ψ *be a subset of* $\mathbb{R}^n$. *Suppose that there exists* $V \in C^1(\mathbb{R}^n;\mathbb{R})$ *such that*

(1) $\langle F(x), V'(x)\rangle + \varphi(x) - \varphi(x - V'(x)) \geqslant 0,\ x \in D(\partial\varphi) \cap \Psi$.

Then $\mathcal{S}(F,\varphi) \cap \Psi \subset E(F,\varphi,V)$.

Proof. Let $z \in \Psi \cap \mathcal{S}(F,\varphi)$ be given. We have $z \in D(\partial\varphi) \cap \Psi$ and

$$\bigl\langle F(z), v - z\bigr\rangle + \varphi(v) - \varphi(z) \geqslant 0, \quad \forall v \in \mathbb{R}^n. \tag{19}$$

Setting $v = z - V'(z)$ in (19), we get

$$\bigl\langle F(z), V'(z)\bigr\rangle + \varphi(z) - \varphi\bigl(z - V'(z)\bigr) \leqslant 0.$$

Then using assumption (1), we obtain:

$$\bigl\langle F(z), V'(z)\bigr\rangle + \varphi(z) - \varphi\bigl(z - V'(z)\bigr) = 0. \qquad \square$$

Proposition 2. *Let the assumptions of Theorem* 1 *together with condition* (17)*. Suppose that there exists* $\sigma > 0$ *and* $V \in C^1(\mathbb{R}^n; \mathbb{R})$ *such that*

(1) $\langle F(x), V'(x)\rangle + \varphi(x) - \varphi(x - V'(x)) \geqslant 0,\ x \in D(\partial\varphi) \cap B_\sigma$;
(2) $E(F, \varphi, V) \cap B_\sigma = \{0\}$.

Then the trivial stationary solution of (9)–(10) *is isolated in* $\mathcal{S}(F, \varphi)$.

Proof. We claim that $B_\sigma \cap \mathcal{S}(F, \varphi) = \{0\}$. Indeed, setting $\Psi := B_\sigma$ and using assumption (2) together with Proposition 1, we obtain:

$$B_\sigma \cap \mathcal{S}(F, \varphi) = \Psi \cap \mathcal{S}(F, \varphi) \subset \Psi \cap E(F, \varphi, V) = \{0\}. \qquad \square$$

The following results can be proved by the same arguments as the ones used in the proof of Propositions 1 and 2.

Proposition 3. *Let the assumptions of Theorem* 1 *together with condition* (17) *hold. Suppose that there exists* $V \in C^1(\mathbb{R}^n; \mathbb{R})$ *such that*

(1) $\langle F(x), V'(x)\rangle + \varphi(x) - \varphi(x - V'(x)) \geqslant 0,\ x \in D(\partial\varphi)$.

Then $\mathcal{S}(F, \varphi) \subset E(F, \varphi, V)$.

Proposition 4. *Let the assumptions of Theorem* 1 *together with condition* (17)*. Suppose that there exists* $V \in C^1(\mathbb{R}^n; \mathbb{R})$ *such that*

(1) $\langle F(x), V'(x)\rangle + \varphi(x) - \varphi(x - V'(x)) \geqslant 0,\ x \in D(\partial\varphi)$;
(2) $E(F, \varphi, V) = \{0\}$.

Then $\mathcal{S}(F, \varphi) = \{0\}$*, i.e., the trivial stationary solution of* (9)–(10) *is the unique stationary solution of* (9)–(10)*.*

3. The invariance theorem

Let the assumptions of Theorem 1 hold. For $x_0 \in D(\partial\varphi)$, we denote by $\gamma(x_0)$ the *orbit*

$$\gamma(x_0) := \bigl\{x(\tau; t_0, x_0)\colon\ \tau \geqslant t_0\bigr\}$$

and by $\Lambda(x_0)$ the *limit set*,

$$\Lambda(x_0) := \bigl\{z \in \mathbb{R}^n\colon \exists \{\tau_i\} \subset [t_0, +\infty);\ \tau_i \to +\infty \text{ and } x(\tau_i; t_0, x_0) \to z\bigr\}.$$

We say that a set $\mathcal{D} \subset D(\partial\varphi)$ is *invariant* provided that

$$x_0 \in \mathcal{D} \Rightarrow \gamma(x_0) \subset \mathcal{D}.$$

Here we denote by $d(s, \mathcal{M})$ the distance from a point $s \in \mathbb{R}^n$ to a set $\mathcal{M} \subset \mathbb{R}^n$, that is $d(s, \mathcal{M}) := \inf_{m\in\mathcal{M}} \|s - m\|$.

Remark 2. Let $x_0 \in D(\partial\varphi)$ be given.

(i) It is clear that

$$\gamma(x_0) \subset D(\partial\varphi), \qquad \Lambda(x_0) \subset \overline{D(\partial\varphi)}.$$

(ii) It is easy to check that

$$\Lambda(x_0) \subset \overline{\gamma(x_0)}.$$

(iii) If $\gamma(x_0)$ is bounded, then $\Lambda(x_0) \neq \emptyset$.

Indeed, if $\gamma(x_0)$ is bounded, then we may find a sequence $x(\tau_i; t_0, x_0)(\tau_i \geqslant t_0)$ such that $x(\tau_i; t_0, x_0) \to z \in \mathbb{R}^n$. It results that $z \in \Lambda(x_0)$.

(iv) If $\gamma(x_0)$ is bounded, then

$$\lim_{\tau\to+\infty} d\bigl(x(\tau; t_0, x_0), \Lambda(x_0)\bigr) = 0.$$

Indeed, if we suppose the contrary then we can find $\varepsilon > 0$ and $\{\tau_i\} \subset [t_0, +\infty)$ such that $\tau_i \to +\infty$ and $d(x(\tau_i; t_0, x_0), \Lambda(x_0)) \geqslant \varepsilon$. The sequence $x(\tau_i; t_0, x_0)$ is bounded and along a subsequence, we may suppose that $x(\tau_i; t_0, x_0) \to x^*$. Thus $x^* \in \Lambda(x_0)$. On the other hand, we get the contradiction $d(x*, \Lambda(x_0)) \geqslant \varepsilon$.

(v) The set of stationary solutions $\mathcal{S}(F, \varphi)$ is invariant. Indeed, if $x_0 \in \mathcal{S}(F, \varphi)$ then $x(\tau; t_0, x_0) = x_0, \forall t \geqslant t_0$, and thus $\gamma(x_0) = \{x_0\} \subset \mathcal{S}(F, \varphi)$.

Thanks to Theorem 2, we can prove that the set $\Lambda(x_0)$ is invariant by using standard topological arguments (see, e.g., [16]).

Theorem 4. *Let the assumptions of Theorem* 1. *Let $x_0 \in D(\partial\varphi)$ be given. The set $\Lambda(x_0)$ is invariant.*

Proof. Let $z \in \Lambda(x_0)$ be given. There exists $\{\tau_i\} \subset [t_0, +\infty)$ such that $\tau_i \to +\infty$ and $x(\tau_i; t_0, x_0) \to z$. Let $\tau \geqslant t_0$ be given. Using Theorem 2, we obtain

$$x(\tau; t_0, z) = \lim_{i\to\infty} x\bigl(\tau; t_0, x(\tau_i; t_0, x_0)\bigr).$$

Then remarking that $x(\tau; t_0, x(\tau_i; t_0, x_0)) = x(\tau - t_0 + \tau_i; t_0, x_0)$, we get $x(\tau; t_0, z) = \lim_{i\to\infty} x(\tau - t_0 + \tau_i; t_0, x_0)$. Thus setting $w_i := \tau - t_0 + \tau_i$, we see that $w_i \geqslant t_0$, $w_i \to +\infty$ and $x(w_i; t_0, x_0) \to x(\tau; t_0, z)$. It results that $x(\tau; t_0, z) \in \Lambda(x_0)$.

Our goal is now to prove an extension of the LaSalle Invariance Principle applicable to the first-order evolution variational inequality given in (4).

Lemma 2. *Let the assumptions of Theorem* 1. *Let Ψ be a compact subset of $\mathbb{R}^n$. We assume that there exists $V \in C^1(\mathbb{R}^n; \mathbb{R})$ such that*

(1) $\langle F(x), V'(x)\rangle + \varphi(x) - \varphi(x - V'(x)) \geqslant 0,\ x \in D(\partial\varphi) \cap \Psi.$

Let $x_0 \in D(\partial\varphi)$ be given. If $\gamma(x_0) \subset \Psi$ then there exists a constant $k \in \mathbb{R}$ such that

$$V(x) = k, \quad \forall x \in \Lambda(x_0).$$

Proof. Let $T > 0$ be given. We define the mapping $V^* : [t_0; +\infty) \to \mathbb{R}$ by the formula

$$V^*(t) := V\bigl(x(t; t_0, x_0)\bigr).$$

The function $x(\cdot) \equiv x(\cdot; t_0, x_0)$ is absolutely continuous on $[t_0, t_0 + T]$ and thus V^* is a.e. strongly differentiable on $[t_0, t_0 + T]$. We have:

$$\frac{\mathrm{d}V^*}{\mathrm{d}t}(t) = \left\langle V'\bigl(x(t)\bigr), \frac{\mathrm{d}x}{\mathrm{d}t}(t)\right\rangle, \quad \text{a.e. } t \in [t_0, t_0 + T].$$

We know (by hypothesis) that

$$x(t) \in D(\partial\varphi) \cap \Psi, \quad \forall t \geqslant t_0,$$

and

$$\left\langle \frac{\mathrm{d}x}{\mathrm{d}t}(t) + F\bigl(x(t)\bigr), v - x(t)\right\rangle + \varphi(v) - \varphi\bigl(x(t)\bigr) \geqslant 0, \quad \forall v \in \mathbb{R}^n,\ \text{a.e. } t \geqslant t_0. \tag{20}$$

Setting $v = x(t) - V'(x(t))$ in (20), we obtain:

$$\left\langle \frac{\mathrm{d}x}{\mathrm{d}t}(t), V'\bigl(x(t)\bigr)\right\rangle \leqslant -\varphi\bigl(x(t)\bigr) + \varphi\bigl(x(t) - V'\bigl(x(t)\bigr)\bigr) - \bigl\langle F\bigl(x(t)\bigr), V'\bigl(x(t)\bigr)\bigr\rangle, \quad \text{a.e. } t \geqslant t_0.$$

and thus using assumption (1), we obtain:

$$\left\langle \frac{\mathrm{d}x}{\mathrm{d}t}(t), V'\big(x(t)\big)\right\rangle \leqslant 0, \quad \text{a.e. } t \geqslant t_0. \tag{21}$$

Thus

$$\frac{\mathrm{d}V^*}{\mathrm{d}t}(t) \leqslant 0, \quad \text{a.e. } t \in [t_0, t_0 + T].$$

We know that $x \in C^0([t_0, t_0 + T]; \mathbb{R}^n)$, $\frac{\mathrm{d}x}{\mathrm{d}t} \in L^\infty(t_0, t_0 + T; \mathbb{R}^n)$ and $V \in C^1(\mathbb{R}^n; \mathbb{R})$. It follows that $V^* \in W^{1,1}(t_0, t_0 + T; \mathbb{R}^n)$ and applying, e.g., Lemma 3.1 in [6], we obtain that V^* is decreasing on $[t_0, t_0 + T]$. The real T has been chosen arbitrary and thus V^* is decreasing on $[t_0, +\infty)$. Moreover V^* is bounded from below on $[t_0, +\infty)$ since $\gamma(x_0) \subset \Psi$ and V is continuous on the compact set Ψ. It results that

$$\lim_{\tau \to +\infty} V\big(x(\tau; t_0, x_0)\big) = k,$$

for some $k \in \mathbb{R}$.

Let $y \in \Lambda(x_0)$ be given. There exists $\{\tau_i\} \subset [t_0, +\infty)$ such that $\tau_i \to +\infty$ and $x(\tau_i; t_0, x_0) \to y$. By continuity

$$\lim_{i \to +\infty} V\big(x(\tau_i; t_0, x_0)\big) = V(y).$$

Therefore $V(y) = k$. Here y has been chosen arbitrary in $\Lambda(x_0)$ and thus

$$V(y) = k, \quad \forall y \in \Lambda(x_0). \qquad \square$$

Lemma 3. *Let the assumptions of Theorem* 1. *We assume that there exists* $V \in C^1(\mathbb{R}^n; \mathbb{R})$ *such that*

(1) $\langle F(x), V'(x)\rangle + \varphi(x) - \varphi(x - V'(x)) \geqslant 0,\ x \in D(\partial\varphi)$.

Let $a \in \mathbb{R}$ *be given and set*:

$$\overline{\Psi} := \big\{x \in \mathbb{R}^n\colon\ V(x) \leqslant a\big\}.$$

The set $D(\partial\varphi) \cap \overline{\Psi}$ *is invariant.*

Proof. Let $x_0 \in D(\partial\varphi) \cap \overline{\Psi}$ be given. Then $x_0 \in D(\partial\varphi)$ and $V(x_0) \leqslant a$. If $\tau \geqslant t_0$ then $x(\tau; t_0, x_0) \in D(\partial\varphi)$ and as in the proof of Lemma 2, we check that $V(x(\cdot; t_0, x_0))$ is decreasing on $[t_0, +\infty)$. Thus

$$V\big(x(\tau; t_0, x_0)\big) \leqslant V\big(x(t_0; t_0, x_0)\big) = V(x_0) \leqslant a.$$

It results that

$$\gamma(x_0) \subset D(\partial\varphi) \cap \overline{\Psi}. \qquad \square$$

Theorem 5 (Invariance theorem). *Suppose that the assumptions of Theorem* 1 *hold. Let* $\Psi \subset \mathbb{R}^n$ *be a compact set and* $V \in C^1(\mathbb{R}^n; \mathbb{R})$ *a function such that*

(1) $\varphi(\cdot) - \varphi(\cdot - V'(\cdot))$ *is lower semicontinuous on* $D(\partial\varphi) \cap \Psi$;
(2) $\langle F(x), V'(x)\rangle + \varphi(x) - \varphi(x - V'(x)) \geqslant 0,\ x \in D(\partial\varphi) \cap \Psi$;
(3) $D(\partial\varphi)$ *is closed.*

We set:

$$E_\Psi(F, \varphi, V) := E(F, \varphi, V) \cap \Psi$$

and we denote by $\mathcal{M}$ *the largest invariant subset of* $E_\Psi(F, \varphi, V)$. *Then, for each* $x_0 \in D(\partial\varphi)$ *such that* $\gamma(x_0) \subset \Psi$, *we have*

$$\lim_{\tau\to+\infty} d\big(x(\tau; t_0, x_0), \mathcal{M}\big) = 0.$$

Proof. Here $\gamma(x_0)$ is bounded and thus (see Remark 2, (iii) and (iv)) $\Lambda(x_0)$ is nonempty and

$$\lim_{\tau\to+\infty} d\big(x(\tau; t_0, x_0), \Lambda(x_0)\big) = 0.$$

Let us now check that $\Lambda(x_0) \subset E_\Psi(F, \varphi, V)$. We first note that

$$\Lambda(x_0) \subset \overline{\gamma(x_0)} \subset \overline{D(\partial\varphi) \cap \Psi} = D(\partial\varphi) \cap \Psi.$$

From Lemma 2, there exists $k \in \mathbb{R}$ such that $V(x) = k,\ \forall x \in \Lambda(x_0)$. Let $z \in \Lambda(x_0)$ be given. Using Theorem 4, we see that $x(t; t_0, z) \in \Lambda(x_0),\ \forall t \geqslant t_0$ and thus

$$V\big(x(t; t_0, z)\big) = k, \quad \forall t \geqslant t_0.$$

It results that

$$\frac{\mathrm{d}}{\mathrm{d}t} V\big(x(t; t_0, z)\big) = 0, \quad \text{a.e. } t \geqslant t_0. \tag{22}$$

Setting $x(\cdot) \equiv x(\cdot; t_0, z)$, we check as in the proof of Lemma 2 that

$$\left\langle V'\big(x(t)\big), \frac{\mathrm{d}x}{\mathrm{d}t}(t)\right\rangle \leqslant -\big\langle F\big(x(t)\big), V'\big(x(t)\big)\big\rangle \tag{23}$$

$$- \varphi\big(x(t)\big) + \varphi\big(x(t) - V'\big(x(t)\big)\big), \quad \text{a.e. } t \geqslant t_0. \tag{24}$$

From (22) and (24) we deduce that

$$\big\langle F\big(x(t)\big), V'\big(x(t)\big)\big\rangle + \varphi\big(x(t)\big) - \varphi\big(x(t) - V'\big(x(t)\big)\big) \leqslant 0, \quad \text{a.e. } t \geqslant t_0.$$

Using assumption (1), we see that the mapping,

$$t \mapsto \big\langle F\big(x(t;t_0,z)\big), V'\big(x(t;t_0,z)\big)\big\rangle + \varphi\big(x(t;t_0,z)\big) - \varphi\big(x(t;t_0,z) - V'\big(x(t;t_0,z)\big)\big)$$

is lower semicontinuous on $[t_0, +\infty)$ and thus taking the lim inf as $t \to t_0$, we obtain:

$$\big\langle F(z), V'(z)\big\rangle + \varphi(z) - \varphi\big(z - V'(z)\big) \leqslant 0.$$

This together with assumption (2) ensure that $z \in E_\Psi(F,\varphi,V)$. Finally $\Lambda(x_0) \subset \mathcal{M}$ since $\Lambda(x_0) \subset E_\Psi(F,\varphi,V)$ and $\Lambda(x_0)$ is invariant (see Theorem 4). The conclusion follows. □

Remark 3. Note that the conditions of Theorem 5 ensure that

$$\mathcal{S}(F,\varphi) \cap \Psi \subset \mathcal{M}.$$

Indeed, Proposition 1 yields $\mathcal{S}(F,\varphi) \cap \Psi \subset E_\Psi(F,\varphi,V)$ and $\mathcal{S}(F,\varphi) \cap \Psi$ is invariant.

Corollary 1. *Suppose that the assumptions of Theorem* 1 *hold. Let $V \in C^1(\mathbb{R}^n;\mathbb{R})$ be a function such that*

(1) $\varphi(\cdot) - \varphi(\cdot - V'(\cdot))$ *is lower semicontinuous on* $D(\partial\varphi)$;
(2) $\langle F(x), V'(x)\rangle + \varphi(x) - \varphi(x - V'(x)) \geqslant 0,\ x \in D(\partial\varphi)$;
(3) $V(x) \to +\infty$ *as* $\|x\| \to +\infty$, $x \in D(\partial\varphi)$;
(4) $D(\partial\varphi)$ is closed.

Let $\mathcal{M}$ be the largest invariant subset of $E(F,\varphi,V)$. Then, for each $x_0 \in D(\partial\varphi)$, the orbit $\gamma(x_0)$ is bounded and

$$\lim_{\tau\to+\infty} d\big(x(\tau;t_0,x_0), \mathcal{M}\big) = 0.$$

Proof. Let $x_0 \in D(\partial\varphi)$ be given. We set $\overline{\Psi} := \{x \in \mathbb{R}^n\colon\ V(x) \leqslant V(x_0)\}$ and $\Psi = \overline{\Psi} \cap D(\partial\varphi)$. The set $\overline{\Psi}$ is closed. Assumptions (3) and (4) ensure that $D(\partial\varphi) \cap \overline{\Psi}$ is bounded and closed. Thus Ψ is compact. Lemma 3 ensures that Ψ is invariant. Here $x_0 \in \Psi$ and thus $\gamma(x_0) \subset \Psi$. It results that $\gamma(x_0)$ is bounded. Moreover, from Theorem 5, we obtain:

$$\lim_{\tau\to+\infty} d\big(x(\tau;t_0,x_0), \mathcal{M}_*\big) = 0,$$

where $\mathcal{M}_*$ is the largest invariant subset of $E_\Psi(F,\varphi,V)$. It is clear that $\mathcal{M}_* \subset \mathcal{M}$ and the conclusion follows. □

Corollary 2. *Let the assumptions of Theorem* 1 *together with condition* (17). *Suppose that there exists* $V \in C^1(\mathbb{R}^n;\mathbb{R})$ *such that*

(1) $V(x) \geqslant a(\|x\|),\ x \in D(\partial\varphi)$, *with* $a:\mathbb{R}_+ \to \mathbb{R}$ *satisfying* $a(0)=0$, a *strictly increasing on* $\mathbb{R}_+$;
(2) $V(0)=0$;
(3) $\varphi(\cdot)-\varphi(\cdot - V'(\cdot))$ *is lower semicontinuous on* $D(\partial\varphi)$;
(4) $\langle F(x), V'(x)\rangle + \varphi(x) - \varphi(x - V'(x)) \geqslant 0,\ x \in D(\partial\varphi)$;
(5) $D(\partial\varphi)$ *is closed*;
(6) $E(F,\varphi,V)=\{0\}$.

Then the trivial solution of (9)–(10) *is*

(a) *the unique stationary solution of* (9)–(10),
(b) *asymptotically stable,*
(c) *globally attractive, i.e., for each* $x_0 \in D(\partial\varphi)$, $\lim_{t\to+\infty}\|x(t;t_0,x_0)\|=0$.

Proof. Assertion (a) is a consequence of Proposition 4. The stability is a direct consequence of Theorem 3. Moreover, we may apply Corollary 1 with $\mathcal{M}=\{0\}$ (since $E(F,\varphi,V)=\{0\}$) to obtain that for any $x_0 \in D(\partial\varphi)$ the limit

$$\lim_{\tau\to+\infty} x(\tau;t_0,x_0)=0$$

holds. Assertions (b) and (c) follow. $\square$

Corollary 3. *Let the assumptions of Theorem* 1 *together with condition* (17). *Suppose that there exists* $\sigma>0$ *and* $V \in C^1(\mathbb{R}^n;\mathbb{R})$ *such that*

(1) $V(x) \geqslant a(\|x\|),\ x \in D(\partial\varphi)\cap B_\sigma$, *with* $a:[0,\sigma]\to\mathbb{R}$ *satisfying* $a(t)>0, \forall t \in (0,\sigma)$;
(2) $V(0)=0$;
(3) $\varphi(\cdot)-\varphi(\cdot - V'(\cdot))$ *is lower semicontinuous on* $D(\partial\varphi)\cap B_\sigma$;
(4) $\langle F(x), V'(x)\rangle + \varphi(x) - \varphi(x - V'(x)) \geqslant 0,\ x \in D(\partial\varphi)\cap B_\sigma$;
(5) $D(\partial\varphi)$ *is closed*;
(6) $E(F,\varphi,V)\cap B_\sigma=\{0\}$.

Then the trivial solution of (9)–(10) *is* (a) *isolated in* $\mathcal{S}(F,\varphi)$, (b) *asymptotically stable.*

Proof. Assertion (a) is a direct consequence of Proposition 2. The stability follows from Theorem 3. The stability ensures the existence of $\delta>0$ such that if $x_0 \in D(\partial\varphi)\cap B_\delta$, then

$$\gamma(x_0)\subset B_\sigma.$$

Applying Theorem 5 with $\Psi = B_\sigma$, we obtain for $x_0 \in D(\partial\varphi)\cap B_\delta$ that

$$\lim_{t\to+\infty} d\big(x(t;t_0,x_0),\mathcal{M}\big)=0,$$

where $\mathcal{M}$ is the largest invariant subset of $E_\Psi(F,\varphi,V)$. It is clear that assumption (6) yields $\mathcal{M}=\{0\}$. The attractivity and assertion (b) follow. □

Corollary 4. *Let the assumptions of Theorem* 1 *together with condition* (17). *Assume that* $D(\partial\varphi)$ *is closed and suppose that there exists* $\sigma>0$ *such that*

$$\big\langle F(x),x\big\rangle+\varphi(x)-\varphi(0)>0,\quad x\in D(\partial\varphi)\cap B_\sigma,\ x\neq 0.$$

Then the trivial stationary solution of (9)–(10) *is* (a) *isolated in* $\mathcal{S}(F,\varphi)$ *and* (b) *asymptotically stable.*

Proof. This follows from Corollary 3 that we may apply with $V\in C^1(\mathbb{R}^n;\mathbb{R})$ defined by $V(x)=1/2\|x\|^2$, $x\in\mathbb{R}^n$. □

4. Second-order dynamic systems

In this section, we deal with the following class of second-order dynamic systems:

Let $\Phi:\mathbb{R}^l\to\mathbb{R}\cup\{+\infty\}$ be a proper convex and lower semicontinuous function. Let $M,C,K\in\mathbb{R}^{m\times m}$, $H_1\in\mathbb{R}^{m\times l}$ and $H_2\in\mathbb{R}^{l\times m}$ be given matrices. For $(t_0,q_0,\dot q_0)\in\mathbb{R}\times\mathbb{R}^m\times\mathbb{R}^m$ with $H_2\dot q_0\in D(\partial\Phi)$, we consider the problem $P(t_0,q_0,\dot q_0)$: Find a function $t\mapsto q(t)$ $(t\geqslant t_0)$ with $q\in C^1([t_0,+\infty);\mathbb{R}^m)$, and such that

$$\frac{\mathrm{d}^2q}{\mathrm{d}t^2}\in L^\infty_{\mathrm{loc}}(t_0,+\infty;\mathbb{R}^m),\tag{25}$$

$$\frac{\mathrm{d}q}{\mathrm{d}t}\text{ is right-differentiable on }[t_0,+\infty),\tag{26}$$

$$q(t_0)=q_0,\tag{27}$$

$$\frac{\mathrm{d}q}{\mathrm{d}t}(t_0)=\dot q_0,\tag{28}$$

$$H_2\frac{\mathrm{d}q}{\mathrm{d}t}(t)\in D(\partial\Phi),\quad t\geqslant t_0,\tag{29}$$

$$M\frac{\mathrm{d}^2q}{\mathrm{d}t^2}(t)+C\frac{\mathrm{d}q}{\mathrm{d}t}(t)+Kq(t)\in -H_1\partial\Phi\left(H_2\frac{\mathrm{d}q}{\mathrm{d}t}(t)\right),\quad \text{a.e. } t\geqslant t_0.\tag{30}$$

The model in (30) can be used in Mechanics to describe the motion of various systems having frictional contact. For such problems, m is the number of degrees of freedom of the system, M is the mass matrix of the system, C is the viscous damping matrix of the system and K is the stiffness matrix. The term $H_1\partial\Phi(H_2.)$ is used to model the unilaterality of the contact induced by friction forces.

The Euclidean scalar product in $\mathbb{R}^m$ is denoted by $\langle\cdot,\cdot\rangle_m$ and the corresponding norm by $\|\cdot\|_m$. The subordinate matrix norm is also denoted by $\|\cdot\|_m$. In this section, we also use the notations I_m and $0_{p\times q}$ to denote the $m\times m$ identity matrix and the $p\times q$ null matrix respectively.

Theorem 6 (Existence and uniqueness). *Suppose that the following assumptions are satisfied*:

(1) *M is nonsingular*;
(2) *there exists a matrix $R\in\mathbb{R}^{m\times m}$, symmetric and nonsingular such that*:

$$R^{-2}H_2^{\mathrm{T}}=M^{-1}H_1;$$

(3) *there exists $y_0=H_2R^{-1}x_0$ ($x_0\in\mathbb{R}^m$), at which Φ is finite and continuous.*

Let $t_0\in\mathbb{R}$, $q_0,\dot{q}_0\in\mathbb{R}^m$ with $H_2\dot{q}_0\in D(\partial\Phi)$. Then there exists a unique $q\in C^1([t_0,+\infty);\mathbb{R}^m)$ satisfying conditions (25)–(30).

Proof. Let us here for a function f use the notations $\ddot{f}=\frac{\mathrm{d}^2f}{\mathrm{d}t^2}$ and $\dot{f}=\frac{\mathrm{d}f}{\mathrm{d}t}$. We first remark that (30), i.e.,

$$M\ddot{q}+C\dot{q}+Kq\in -H_1\partial\Phi(H_2\dot{q})$$

is equivalent to

$$\ddot{q}+M^{-1}C\dot{q}+M^{-1}Kq\in -M^{-1}H_1\partial\Phi(H_2\dot{q}).$$

Hence,

$$R\ddot{q}+RM^{-1}CR^{-1}R\dot{q}+RM^{-1}KR^{-1}Rq\in -RM^{-1}H_1\partial\Phi\big(H_2R^{-1}R\dot{q}\big). \tag{31}$$

Setting $z=Rq$ in (31), we get:

$$\ddot{z}+RM^{-1}CR^{-1}\dot{z}+RM^{-1}KR^{-1}z\in -RM^{-1}H_1\partial\Phi\big(H_2R^{-1}\dot{z}\big).$$

Using now assumption (2), we obtain:

$$\ddot{z}+RM^{-1}CR^{-1}\dot{z}+RM^{-1}KR^{-1}z\in -R^{-1}H_2^{\mathrm{T}}\partial\Phi\big(H_2R^{-1}\dot{z}\big). \tag{32}$$

Let us here define the function $\chi:\mathbb{R}^m\to\mathbb{R}\cup\{+\infty\}$ by the formula

$$\chi(w)=\big(\Phi\circ H_2R^{-1}\big)(w),\quad \forall w\in\mathbb{R}^m.$$

It is clear that χ is convex and lower semicontinuous. Moreover, thanks to assumption (3), χ is proper and we have (see, e.g., Proposition 2.4.5 in [13]):

$$\partial\chi(w)=R^{-1}H_2^{\mathrm{T}}\partial\Phi\big(H_2R^{-1}w\big),\quad \forall w\in\mathbb{R}^m.$$

Thus (32) reduces to

$$\ddot{z} + RM^{-1}CR^{-1}\dot{z} + RM^{-1}KR^{-1}z \in -\partial\chi(\dot{z}). \tag{33}$$

We note also that (27), (28) and (29) can be written here respectively in term of the variable z as $z(t_0) = Rq_0$, $\dot{z}(t_0) = R\dot{q}_0$ and $\dot{z}(t) \in D(\partial\chi), \forall t \geqslant t_0$. Moreover, $R\dot{q}_0 \in D(\partial\chi)$ since $H_2\dot{q}_0 \in D(\partial\Phi)$. Let us now set:

$$x_1 := z, \quad x_2 := \dot{z}, \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \tag{34}$$

It is clear that (33) is equivalent to the following first-order system:

$$\begin{cases} \dot{x}_1 - x_2 = 0, \\ \dot{x}_2 + RM^{-1}CR^{-1}x_2 + RM^{-1}KR^{-1}x_1 \in -\partial\chi(x_2). \end{cases}$$

It results that problem $P(t_0, q_0, \dot{q}_0)$ can be written as follows:

$$\begin{cases} \dot{x} + Ax \in -\partial\varphi(x), \\ x(t_0) = x_0, \end{cases}$$

where the matrix $A \in \mathbb{R}^{n\times n}$ $(n = 2m)$ is defined by:

$$A = \begin{pmatrix} 0_{m\times m} & -I_m \\ RM^{-1}KR^{-1} & RM^{-1}CR^{-1} \end{pmatrix}, \tag{35}$$

the vector $x_0 \in \mathbb{R}^n$ is given by:

$$x_0 = \begin{pmatrix} Rq_0 \\ R\dot{q}_0 \end{pmatrix}, \tag{36}$$

and the proper, convex and lower semicontinuous function $\varphi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is defined by:

$$\varphi(x) := \chi(x_2). \tag{37}$$

The result is thus a direct consequence of Theorem 1 (with $F(\cdot) = A$). Indeed, A is Lipschitz continuous (see Remark 1). $\square$

Let the assumptions of Theorem 6 and let us now denote by $q(\cdot; t_0, q_0, \dot{q}_0)$ the unique solution of problem $P(t_0, q_0, \dot{q}_0)$.

The set $\mathcal{W}$ of stationary solutions of (29)–(30) is given by:

$$\mathcal{W} = \{\bar{q} \in \mathbb{R}^m:\ K\bar{q} \in -H_1\partial\Phi(0)\}.$$

We suppose that

$$0 \in D(\partial \Phi). \tag{38}$$

Remark 4. (i) If $0 \in \partial\Phi(0)$ then it is clear that $0 \in \mathcal{W}$;

(ii) if $0 \in D(\partial\Phi)$ and K is nonsingular then $\mathcal{W} = -K^{-1}H_1\partial\Phi(0)$;

(iii) if $\partial\Phi(0) = \{0\}$ then $\mathcal{W} = \ker K$;

(iv) If $\Phi'(0)$ exists and K is nonsingular then the trivial stationary solution of (29)–(30) is the unique stationary solution of (29)–(30). Indeed, here we have $\mathcal{W} = \{-K^{-1}H_1\Phi'(0)\}$.

We consider the stability of a stationary solution with respect to the "generalized coordinates" $q_1, \ldots, q_m$ and the "generalized velocities" $\frac{\mathrm{d}q_1}{\mathrm{d}t}, \ldots, \frac{\mathrm{d}q_m}{\mathrm{d}t}$. More precisely, we say that a stationary solution $\bar{q} \in \mathcal{W}$ is *stable* provided that for any $\varepsilon > 0$ there exists $\eta(\varepsilon) > 0$ such that for any $q_0 \in \mathbb{R}^m$, $\dot{q}_0 \in \mathbb{R}^m$, $H_2\dot{q}_0 \in D(\partial\Phi)$ with $\sqrt{\|q_0 - \bar{q}\|_m^2 + \|\dot{q}_0\|_m^2} \leqslant \eta$ the solution $q(\cdot; t_0, q_0, \dot{q}_0)$ of problem $P(t_0, q_0, \dot{q}_0)$ satisfies

$$\sqrt{\left\|q(t; t_0, q_0, \dot{q}_0) - \bar{q}\right\|_m^2 + \left\|\frac{\mathrm{d}q}{\mathrm{d}t}(t; t_0, q_0, \dot{q}_0)\right\|_m^2} \leqslant \varepsilon, \quad \forall t \geqslant t_0.$$

If there exists a $\delta > 0$ such that for any $q_0 \in \mathbb{R}^m$, $\dot{q}_0 \in \mathbb{R}^m$, $H_2\dot{q}_0 \in D(\partial\Phi)$ with $\sqrt{\|q_0 - \bar{q}\|_m^2 + \|\dot{q}_0\|_m^2} \leqslant \delta$ the solution $q(\cdot; t_0, q_0, \dot{q}_0)$ of problem $P(t_0, q_0, \dot{q}_0)$ satisfies the limits:

$$\lim_{t \to +\infty} \left\|q(t; t_0, q_0, \dot{q}_0) - \bar{q}\right\|_m = 0 \tag{39}$$

and

$$\lim_{t \to +\infty} \left\|\frac{\mathrm{d}q}{\mathrm{d}t}(t; t_0, q_0, \dot{q}_0)\right\|_m = 0, \tag{40}$$

then we say that the stationary solution $\bar{q}$ is *attractive*. If the limits in (39) and (40) hold for any $q_0 \in \mathbb{R}^m$, $\dot{q}_0 \in \mathbb{R}^m$, $H_2\dot{q}_0 \in D(\partial\Phi)$ then we say that the stationary solution $\bar{q}$ is *globally attractive*. Finally, a stable and attractive stationary solution is said *asymptotically stable*.

Theorem 7 (Stability). *Let the assumptions of Theorem* 6 *together with condition* (38). *Suppose in addition that*

(1) $RM^{-1}CR^{-1}$ *is positive semidefinite*;
(2) $RM^{-1}KR^{-1}$ *is symmetric and positive definite.*

Then $\mathcal{W} \neq \emptyset$ and any stationary solution $\bar{q} \in \mathcal{W}$ of (29)–(30) *is stable.*

Proof. Condition (38) ensures that $\partial\Phi(0) \neq \emptyset$ and assumption (2) entails that K is nonsingular. Thus $\mathcal{W} = -K^{-1}H_1\partial\Phi(0) \neq \emptyset$.

Let $\bar{q} \in \mathcal{W}$ be given. Setting $Q := q - \bar{q}$, we see that the question of stability of $\bar{q}$ reduces to the one of the trivial stationary solution of the system:

$$M\ddot{Q} + C\dot{Q} + KQ + K\bar{q} \in -H_1\partial\Phi(H_2\dot{Q}). \tag{41}$$

Setting $x_1 := RQ$, $x_2 := R\dot{Q}$ and $x := (x_1\ x_2)^{\mathrm{T}}$, we check as in the proof of Theorem 6 that the system in (41) can be written as follows:

$$\dot{x} + F(x) \in -\partial\varphi(x)$$

where

$$F(x) = Ax + \overline{F}, \quad A = \begin{pmatrix} 0_{m\times m} & -I_m \\ RM^{-1}KR^{-1} & RM^{-1}CR^{-1} \end{pmatrix}, \ \overline{F} = \begin{pmatrix} 0_{m\times 1} \\ RM^{-1}K\bar{q} \end{pmatrix},$$

$$\varphi(x) = \chi(x_2)\ \big(:= \Phi \circ H_2R^{-1}(x_2)\big) \quad \text{and} \quad \partial\varphi(x) = \begin{pmatrix} 0_{m\times 1} \\ R^{-1}H_2^{\mathrm{T}}\partial\Phi(H_2R^{-1}x_2) \end{pmatrix}.$$

The mapping $F(\cdot)$ is Lipschitz continuous. Moreover, condition (17) holds since $\bar{q} \in \mathcal{W} \Leftrightarrow K\bar{q} \in -H_1\partial\Phi(0) \Leftrightarrow RM^{-1}K\bar{q} \in -RM^{-1}H_1\partial\Phi(0) \Leftrightarrow RM^{-1}K\bar{q} \in -R^{-1}H_2^{\mathrm{T}}\partial\Phi(0) \Leftrightarrow RM^{-1}K\bar{q} \in -\partial\chi(0) \Leftrightarrow \overline{F} \in -\partial\varphi(0)$.

Let us now check that all the assumptions of Theorem 3 are satisfied. Let $V \in C^1(\mathbb{R}^n; \mathbb{R})$ $(n = 2m)$ be given by

$$V(x) = \frac{1}{2}\big\langle RM^{-1}KR^{-1}x_1, x_1\big\rangle_m + \frac{1}{2}\|x_2\|_m^2.$$

It is clear from hypothesis (2) that assumption (1) of Theorem 3 is satisfied. Assumption (2) of Theorem 3 is also clearly satisfied.

We have:

$$V'(x) = \begin{pmatrix} RM^{-1}KR^{-1}x_1 \\ x_2 \end{pmatrix}.$$

Thus

$$\begin{aligned} &\big\langle Ax, V'(x)\big\rangle + \big\langle \overline{F}, V'(x)\big\rangle + \varphi(x) - \varphi\big(x - V'(x)\big) \\ &\quad = \big\langle RM^{-1}CR^{-1}x_2, x_2\big\rangle_m + \big\langle RM^{-1}K\bar{q}, x_2\big\rangle_m + \chi(x_2) - \chi(0). \end{aligned}$$

Assumption (1) yields

$$\big\langle RM^{-1}CR^{-1}x_2, x_2\big\rangle_m \geqslant 0. \tag{42}$$

Moreover, $\bar{q} \in \mathcal{W}$ and thus $RM^{-1}K\bar{q} \in -\partial\chi(0)$. It results that

$$\big\langle RM^{-1}K\bar{q}, x_2\big\rangle + \chi(x_2) - \chi(0) \geqslant 0. \tag{43}$$

The inequalities in (42) and (43) ensure that hypothesis (3) of Theorem 3 is satisfied. The conclusion is thus a consequence of Theorem 3. □

It is easy to see from the proof of Theorem 7 that the following variant can also be stated.

Theorem 8. *Let the assumptions of Theorem* 6 *together with condition* (38)*. Let $\bar{q} \in \mathcal{W}$ be a stationary solution of* (29)–(30)*. Suppose that*

(1) $\langle RM^{-1}CR^{-1}z + RM^{-1}K\bar{q}, z\rangle_m + \Phi(H_2R^{-1}z) - \Phi(0) \geqslant 0, z \in \mathbb{R}^m$;
(2) $RM^{-1}KR^{-1}$ *is symmetric and positive definite.*

Then $\bar{q}$ is stable.

It follows from Remark 4 that an equilibrium point $\bar{q}$ is in general not isolated in $\mathcal{W}$. The concept of attractivity is for such case not really appropriated. It is then worthwhile to verify if the trajectories of the perturbed solutions are attracted by $\mathcal{W}$.

Theorem 9 (Attractivity of $\mathcal{W}$). *Let the assumptions of Theorem* 6 *together with condition* (38)*. Suppose also that*

(1) $RM^{-1}KR^{-1}$ *is symmetric and positive definite*;
(2) $\langle RM^{-1}CR^{-1}z, z\rangle_m + \Phi(H_2R^{-1}z) - \Phi(0) > 0, z \in \mathbb{R}^m\backslash\{0\}$;
(3) $D(\partial\Phi)$ *is closed.*

Then (a) *for any $q_0 \in \mathbb{R}^m$, $\dot{q}_0 \in \mathbb{R}^m$, $H_2\dot{q}_0 \in D(\partial\Phi)$, the orbit*

$$\Omega(q_0, \dot{q}_0) := \left\{ \left(q(\tau; t_0, q_0, \dot{q}_0)\ \frac{\mathrm{d}q}{\mathrm{d}t}(\tau; t_0, q_0, \dot{q}_0) \right)^{\mathrm{T}} :\ \tau \geqslant t_0 \right\}$$

is bounded and (b) *the following asymptotic properties hold*:

$$\lim_{\tau \to +\infty} d\big(q(\tau; t_0, q_0, \dot{q}_0), \mathcal{W}\big) = 0 \quad \textit{and} \quad \lim_{\tau \to +\infty} \frac{\mathrm{d}q}{\mathrm{d}t}(\tau; t_0, q_0, \dot{q}_0) = 0.$$

Proof. From the proof of Theorem 6, we know that the study of our problem reduces to the one of the first-order system

$$\dot{x} + Ax \in -\partial\varphi(x),$$

where A is defined in (35) and φ is given by (37).

Let us first check that all assumptions of Corollary 1 are satisfied with $V \in C^1(\mathbb{R}^n; \mathbb{R})$ $(n = 2m)$, defined as in the proof of Theorem 7, i.e.,

$$V(x) = \frac{1}{2}\big\langle RM^{-1}KR^{-1}x_1, x_1\big\rangle_m + \frac{1}{2}\|x_2\|_m^2.$$

We have $\varphi(x) - \varphi(x - V'(x)) = \chi(x_2) - \chi(0)$ and the application $x \mapsto \varphi(x) - \varphi(x - V'(x))$ is thus lower semicontinuous. It results that hypothesis (1) of Corollary 1 is satisfied.

We have

$$\langle Ax, V'(x)\rangle + \varphi(x) - \varphi\big(x - V'(x)\big) = \langle RM^{-1}CR^{-1}x_2, x_2\rangle_m + \chi(x_2) - \chi(0).$$

Assumption (2) ensure that hypothesis (2) of Corollary 1 holds.

It is clear that hypothesis (3) of Corollary 1 is satisfied. Finally hypothesis (4) of Corollary 1 follows from assumption (3) which ensures that $D(\partial\varphi) = \mathbb{R}^m \times D(\partial(\Phi \circ H_2R^{-1}))$ is closed.

Here, we have:

$$E(A, \varphi, V) = \big\{x \in D(\partial\varphi)\colon \langle RM^{-1}CR^{-1}x_2, x_2\rangle_m + \chi(x_2) - \chi(0) = 0\big\}.$$

Using assumption (2), we get:

$$E(A, \varphi, V) = \big\{(x_1, 0)\colon\ x_1 \in \mathbb{R}^m\big\}. \tag{44}$$

Corollary 1 ensures that for any $x_0 \in D(\partial\varphi)$, the orbit $\gamma(x_0)$ is bounded. If $q_0 \in \mathbb{R}^m$, $\dot{q}_0 \in \mathbb{R}^m$, $H_2\dot{q}_0 \in D(\partial\Phi)$ then $R\dot{q}_0 \in D(\partial\chi)$. It results that the conclusion of Corollary 1 with $x_0 = (Rq_0\ R\dot{q}_0)^{\mathrm{T}}$ means that the set $\Omega(q_0, \dot{q}_0)$ is bounded. This gives part (a) of our result.

Corollary 1 ensures also that

$$\lim_{\tau \to +\infty} d\big(x(\tau; t_0, x_0), \mathcal{M}\big) = 0,$$

where $\mathcal{M}$ is the largest invariant subset of $E(A, \varphi, V)$. We may apply Proposition 3 to see that $S(A, \varphi) \subset E(A, \varphi, V)$. From Remark 2(v), we know also that $S(A, \varphi)$ is invariant. Thus $S(A, \varphi)$ is an invariant subset of $E(A, \varphi, V)$. We prove now that $S(A, \varphi)$ is the largest invariant subset of $E(A, \varphi, V)$.

Since, $S(A, \varphi) \subset E(A, \varphi, V)$, by (44) we have:

$$S(A, \varphi) = \big\{(x_1, 0)\colon \langle RM^{-1}KR^{-1}x_1, h\rangle_m + \chi(h) - \chi(0) \geqslant 0,\ \forall h \in \mathbb{R}^m\big\}.$$

Let us set

$$\mathcal{N} := \big\{x_1 \in \mathbb{R}^m\colon\ RM^{-1}KR^{-1}x_1 \in -\partial\chi(0)\big\}.$$

Then, we may write

$$S(A, \varphi) = \mathcal{N} \times \{0\}.$$

Let $\mathcal{D}$ be any invariant subset of $E(A,\varphi,V)$ and let $z \in \mathcal{D}$ be given. The function $x(\cdot;t_0,z)$ satisfies:

$$\left\langle \frac{\mathrm{d}x_1}{\mathrm{d}t}(t;t_0,z) - x_2(t;t_0,z), v_1 - x_1(t;t_0,z) \right\rangle_m \geqslant 0, \quad \forall v_1 \in \mathbb{R}^m, \text{ a.e. } t \geqslant t_0, \tag{45}$$

and

$$\left\langle \frac{\mathrm{d}x_2}{\mathrm{d}t}(t;t_0,z) + RM^{-1}KR^{-1}x_1(t;t_0,z) + RM^{-1}CR^{-1}x_2(t;t_0,z), v_2 - x_2(t;t_0,z) \right\rangle_m \\ + \chi(v_2) - \chi\big(x_2(t;t_0,z)\big) \geqslant 0, \quad \forall v_2 \in \mathbb{R}^m, \text{ a.e. } t \geqslant t_0. \tag{46}$$

However, $\gamma(z) \subset \mathcal{D} \subset E(A,\varphi,V)$ and thus $x_2(t;t_0,z) = 0, \forall t \geqslant t_0$. Thus (45) reduces to $\frac{\mathrm{d}x_1}{\mathrm{d}t}(t;t_0,z) = 0$, a.e. $t \geqslant t_0$ from which we deduce that $x_1(\cdot;t_0,z) = z_1,\ \forall t \geqslant t_0$. Then (46) yields

$$\big\langle RM^{-1}KR^{-1}z_1, v_2\big\rangle_m + \chi(v_2) - \chi(0) \geqslant 0, \quad \forall v_2 \in \mathbb{R}^m.$$

Thus

$$z = (z_1, z_2) \in \mathcal{N} \times \{0\}.$$

It results that $\mathcal{D} \subset S(A,\varphi)$ and $S(A,\varphi)$ is well the largest invariant subset of $E(A,\varphi,V)$.

Thus

$$\lim_{\tau\to+\infty} d\big(x(\tau;t_0,x_0), S(A,\varphi)\big) = 0.$$

This implies that

$$\lim_{\tau\to+\infty} d\big(x_1(\tau;t_0,x_0), \mathcal{N}\big) = 0 \tag{47}$$

and

$$\lim_{\tau\to+\infty} x_2(\tau;t_0,x_0) = 0. \tag{48}$$

Recall that in terms of the vector $q = R^{-1}x_1$ of "generalized coordinates" and the vector $\dot{q} = R^{-1}x_2$ of "generalized velocities" we have $RM^{-1}KR^{-1}x_1 \in -\partial\chi(0) \Leftrightarrow Kq \in -MR^{-2}H_2^{\mathrm{T}}\partial\Phi(0) = -H_1\partial\Phi(0)$. Thus the limit in (47) reads

$$\lim_{\tau\to+\infty} d\big(q(\tau;t_0,q_0,\dot{q}_0), \mathcal{W}\big) = 0. \tag{49}$$

On the other hand, the limit in (48) gives:

$$\lim_{\tau\to+\infty} \dot{q}(\tau;t_0,q_0,\dot{q}_0) = 0. \tag{50}$$

Part (b) of our result is thus proved. $\square$

Remark 5. (i) Note that if $0 \in \partial\Phi(0)$ then assumption (2) in Theorem 9 is satisfied provided that either $RM^{-1}CR^{-1}$ is positive definite or $RM^{-1}CR^{-1}$ is positive semidefinite and $\{z \in \mathbb{R}^m : \Phi(H_2R^{-1}z) = \Phi(0)\} = \{0\}$.

(ii) If $H_2^{\mathrm{T}} = H_1$ and M is symmetric and positive definite then the matrix $R = M^{1/2}$ satisfies assumption (2) of Theorem 6. Then conditions (1) and (2) in Theorem 7 hold if and only if C is positive semidefinite and K is symmetric and positive definite. Indeed, here $\langle RM^{-1}CR^{-1}\cdot, \cdot\rangle_m = \langle CM^{-1/2}\cdot, M^{-1/2}\cdot\rangle_m$ and

$$\left\langle RM^{-1}KR^{-1}\cdot, \cdot\right\rangle_m = \left\langle KM^{-1/2}\cdot, M^{-1/2}\cdot\right\rangle_m.$$

(iii) The conditions discussed in Remark 5(ii) are usually satisfied as soon as concrete applications in Mechanics are considered.

(iv) Assumption (1) in Theorem 9 implies that K is nonsingular. Hence $\mathcal{W} = -K^{-1}H_1\partial\Phi(0)$.

(v) Suppose that the assumptions of Theorem 9 hold. Suppose in addition that $\partial\Phi(0) = \{0\}$. Then $\mathcal{W} = \{0\}$ and thus the trivial solution of (29)–(30) is (a) the unique stationary solution of (29)–(30), (b) stable and (c) globally attractive. In particular, the results in (b) and (c) ensure that the trivial solution of (29)–(30) is asymptotically stable

5. Nonsmooth conservative systems

In this section, we consider a mechanical system whose state can be described by m generalized independent coordinates $q = (q_1 \dots q_m)^{\mathrm{T}}$. The kinetic energy of the system is:

$$T = \frac{1}{2}\left\langle M\frac{\mathrm{d}q}{\mathrm{d}t}, \frac{\mathrm{d}q}{\mathrm{d}t}\right\rangle_m,$$

where $M \in \mathbb{R}^{m\times m}$ is symmetric and positive definite. The generalized forces are denoted by Q. We suppose that

$$Q = Q_1 + Q_2,$$

where Q_1 are conservative forces, i.e.,

$$Q_1 = -\Pi'(q)$$

with $\Pi \in C^1(\mathbb{R}^m; \mathbb{R})$ denoting the potential energy of the system and

$$Q_2 \in -H_1\partial\Phi\left(H_1^{\mathrm{T}}\frac{\mathrm{d}q}{\mathrm{d}t}\right),$$

where $\Phi : \mathbb{R}^l \to \mathbb{R} \cup \{+\infty\}$ is a proper convex and lower semicontinuous function and $H_1 \in \mathbb{R}^{m\times l}$ is a given matrix.

The motion of the system is governed by the second-order Lagrange equations:

$$M\frac{\mathrm{d}^2q}{\mathrm{d}t^2}(t)+\Pi'\big(q(t)\big)\in -H_1\partial\Phi\left(H_1^{\mathrm{T}}\frac{\mathrm{d}q}{\mathrm{d}t}(t)\right).$$

More precisely, for $(t_0,q_0,\dot q_0)\in\mathbb{R}\times\mathbb{R}^m\times\mathbb{R}^m$ with $H_1^{\mathrm{T}}\dot q_0\in D(\partial\Phi)$, we consider the problem $L(t_0,q_0,\dot q_0)$: Find a function $t\to q(t)$ $(t\geqslant t_0)$ with $q\in C^1([t_0,+\infty);\mathbb{R}^m)$, and such that

$$\frac{\mathrm{d}^2q}{\mathrm{d}t^2}\in L^\infty_{\mathrm{loc}}(t_0,+\infty;\mathbb{R}^m), \tag{51}$$

$$\frac{\mathrm{d}q}{\mathrm{d}t}\ \text{is right-differentiable on } [t_0,+\infty), \tag{52}$$

$$q(t_0)=q_0, \tag{53}$$

$$\frac{\mathrm{d}q}{\mathrm{d}t}(t_0)=\dot q_0, \tag{54}$$

$$H_1^{\mathrm{T}}\frac{\mathrm{d}q}{\mathrm{d}t}(t)\in D(\partial\Phi),\quad t\geqslant t_0, \tag{55}$$

$$M\frac{\mathrm{d}^2q}{\mathrm{d}t^2}(t)+\Pi'\big(q(t)\big)\in -H_1\partial\Phi\left(H_1^{\mathrm{T}}\frac{\mathrm{d}q}{\mathrm{d}t}(t)\right),\quad \text{a.e. } t\geqslant t_0. \tag{56}$$

Let us first show the existence and uniqueness of solution to problem (51)–(56).

Theorem 10 (Existence and uniqueness). *Let the following assumptions satisfied*:

(1) *M is symmetric and positive definite*;
(2) *Π' is Lipschitz continuous*;
(3) *there exists $y_0=H_1^{\mathrm{T}}M^{-1/2}x_0$ $(x_0\in\mathbb{R}^m)$, at which Φ is finite and continuous.*

Let $t_0\in\mathbb{R}$, $q_0,\dot q_0\in\mathbb{R}^m$ with $H_1^{\mathrm{T}}\dot q_0\in D(\partial\Phi)$. Then there exists a unique $q\in C^1([t_0,+\infty);\mathbb{R}^m)$ satisfying conditions (51)–(56)*.*

Proof. Setting $x=(x_1\ x_2)^{\mathrm{T}}$ with $x_1=Rq$, $x_2=R\dot q$ and $R=M^{1/2}$, we see as in the proof of Theorem 6 that problem $L(t_0,q_0,\dot q_0)$ can be written as follows:

$$\begin{cases}\dot x+F(x)\in-\partial\varphi(x),\\ x(t_0)=x_0,\end{cases} \tag{57}$$

with

$$F(x)=Ax+\overline{F}(x),$$

where the matrix $A \in \mathbb{R}^{n\times n}$ $(n = 2m)$ is defined by:

$$A = \begin{pmatrix} 0_{m\times m} & -I_m \\ 0_{m\times m} & 0_{m\times m} \end{pmatrix}, \tag{58}$$

the mapping $\overline{F} : \mathbb{R}^n \to \mathbb{R}^n$ is defined by:

$$\overline{F}(x) = \begin{pmatrix} 0_{m\times 1} \\ M^{-1/2}\Pi'(M^{-1/2}x_1) \end{pmatrix}, \tag{59}$$

the vector $x_0 \in \mathbb{R}^n$ is given by:

$$x_0 = \begin{pmatrix} M^{1/2}q_0 \\ M^{1/2}\dot{q}_0 \end{pmatrix}, \tag{60}$$

and the proper, convex and lower semicontinuous function $\varphi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is defined by:

$$\varphi(x) := \Phi \circ H_1^{\mathrm{T}} M^{-1/2}(x_2). \tag{61}$$

Here F is Lipschitz continuous. The result is thus a direct consequence of Theorem 1. □

Suppose now that the conditions of Theorem 10 are satisfied and denote by $q(\cdot; t_0, q_0, \dot{q}_0)$ the unique solution of problem $L(t_0, q_0, \dot{q}_0)$.

The set $\mathcal{X}$ of stationary solutions of (55)–(56) is given by:

$$\mathcal{X} = \left\{\bar{q} \in \mathbb{R}^m\colon\ \Pi'(\bar{q}) \in -H_1\partial\Phi(0)\right\}.$$

Suppose that

$$\Pi'(0) = 0, \quad 0 \in \partial\Phi(0). \tag{62}$$

Then $0 \in \mathcal{X}$.

The following result ensures that if at the position of the trivial equilibrium the potential energy has a (strict) local minimum then this trivial equilibrium is stable.

Theorem 11 (Stability). *Let the assumptions of Theorem* 10 *together with condition* (62). *Suppose in addition that there exists* $\bar{\sigma} > 0$ *such that*

(1) $\Pi(0) = 0$;
(2) $\Pi(x) > 0$, $\|x\|_m \leqslant \bar{\sigma}$, $x \neq 0$.

Then the trivial stationary solution of (29)–(30) *is stable.*

Proof. We consider the first-order problem in (57) and we define the function $V \in C^1(\mathbb{R}^n; \mathbb{R})$ $(n = 2m)$ by setting:

$$V(x) = \Pi\big(M^{-1/2}x_1\big) + \frac{1}{2}\|x_2\|_m^2.$$

It is clear from assumption (1) that $V(0) = 0$. Moreover, setting $\sigma := \bar{\sigma}/\|M^{-1/2}\|_m$, using assumption (2) and recalling that $M^{-1/2}$ is nonsingular, we see that

$$V(x) > 0, \quad x \in B_\sigma,\ x \neq 0.$$

Then using a standard result concerning positive definite functions (see, e.g., criterion 3.6 in [16]), we obtain the existence of a continuous and strictly increasing function $a : [0, \sigma] \to \mathbb{R}; t \mapsto a(t)$ such that $a(0) = 0$ and $V(x) \geqslant a(\|x\|), x \in B_\sigma$.

We have:

$$V'(x) = \begin{pmatrix} M^{-1/2}\Pi'\big(M^{-1/2}x_1\big) \\ x_2 \end{pmatrix}$$

and

$$\begin{aligned}\big\langle F(x), V'(x)\big\rangle &= \big\langle Ax + \overline{F}(x), V'(x)\big\rangle + \varphi(x) - \varphi\big(x - V'(x)\big)\\ &= -\big\langle x_2, M^{-1/2}\Pi'\big(M^{-1/2}x_1\big)\big\rangle_m + \big\langle M^{-1/2}\Pi'\big(M^{-1/2}x_1\big), x_2\big\rangle_m\\ &\quad + \Phi\big(H_1^{\mathrm{T}}M^{-1/2}x_2\big) - \Phi(0)\\ &= \Phi\big(H_1^{\mathrm{T}}M^{-1/2}x_2\big) - \Phi(0).\end{aligned}$$

Condition (62) ensures that $\Phi(H_1^{\mathrm{T}}M^{-1/2}x_2) \geqslant \Phi(0)$ and thus $\langle F(x), V'(x)\rangle + \varphi(x) - \varphi(x - V'(x)) \geqslant 0$.

All the assumptions of Theorem 3 are satisfied and the conclusion follows. □

The next result shows that in some particular but important cases we can ensure the asymptotic stability of the trivial stationary solution.

Theorem 12 (Asymptotic stability). *Let the assumptions of Theorem* 6. *Suppose in addition that there exists* $\bar{\sigma} > 0$ *such that*

(1) $\partial\Phi(0) = \{0\}$;
(2) $\Pi'(0) = 0$;
(3) $\Pi(0) = 0$;
(4) $\Pi(x) > 0$, $\|x\|_m \leqslant \bar{\sigma}$, $x \neq 0$;
(5) $\Pi'(x) \neq 0$, $\|x\|_m \leqslant \bar{\sigma}$, $x \neq 0$;
(6) $\Phi(H_1^{\mathrm{T}}x) > \Phi(0)$, $\|x\|_m \leqslant \bar{\sigma}$, $x \neq 0$;
(7) $D(\partial\Phi)$ *is closed.*

Then the trivial stationary solution of (29)–(30) *is* (a) *isolated in* $\mathcal{X}$ *and* (b) *asymptotically stable.*

Proof. (a) It is clear that assumptions (1), (2) and (5) imply that $\mathcal{X} \cap \{\bar{q} \in \mathbb{R}^m\colon \|\bar{q}\|_m \leqslant \bar{\sigma}\} = \{0\}$.

(b) We know that the study of our problem reduces to the one of the first-order system in (57). As in Theorem 11, we consider the function $V \in C^1(\mathbb{R}^n; \mathbb{R})$ $(n = 2m)$ given by:

$$V(x) = \Pi\big(M^{-1/2}x_1\big) + \frac{1}{2}\|x_2\|_m^2.$$

All the assumptions of Theorem 11 are here satisfied and the stability of the trivial stationary solution of the system $\dot{x} + F(x) \in -\partial\varphi(x)$ is ensured. Let $\sigma := \bar{\sigma}/\|M^{-1/2}\|_m$. From Definition 1, there exists $\delta > 0$ such that if $x_0 \in D(\partial\varphi) \cap B_\delta$ then $\gamma(x_0) \subset B_\sigma$.

Let us first check that all assumptions of the Invariance Theorem 5 are satisfied with the compact set $\Psi := B_\sigma$. Indeed, the application

$$x \mapsto \varphi(x) - \varphi\big(x - V'(x)\big) = \Phi\big(H_1^{\mathrm{T}} M^{-1/2} x_2\big) - \Phi(0)$$

is lower semicontinuous, hypothesis (7) ensures that $D(\partial\varphi)$ is closed and we have seen in the proof of Theorem 11 that $\langle F(\cdot), V'(\cdot)\rangle + \varphi(\cdot) - \varphi(\cdot - V'(\cdot)) \geqslant 0$.

Theorem 5 ensures that for $x_0 \in D(\partial\varphi) \cap B_\delta$, we have:

$$\lim_{\tau \to +\infty} d\big(x(\tau; t_0, x_0), \mathcal{M}\big) = 0,$$

where $\mathcal{M}$ is the largest invariant subset of $E_\Psi(F, \varphi, V)$.

Using assumption (6), we obtain:

$$\begin{aligned} E_\Psi(F, \varphi, V) &= \big\{x \in D(\partial\varphi) \cap \Psi\colon\ \Phi\big(H_1^{\mathrm{T}} M^{-1/2} x_2\big) = \Phi(0)\big\} \\ &= \big\{(x_1, 0)\colon\ x_1 \in \mathbb{R}^m,\ \|x_1\| \leqslant \sigma\big\}. \end{aligned}$$

Let $\mathcal{D}$ be any invariant subset of $E_\Psi(F, \varphi, V)$ and let $z \in \mathcal{D}$ be given. The function $x(\cdot; t_0, z)$ satisfies:

$$\frac{\mathrm{d}x_1}{\mathrm{d}t}(t; t_0, z) = x_2(t; t_0, z), \quad \text{a.e. } t \geqslant t_0, \tag{63}$$

and

$$\begin{aligned} &\left\langle \frac{\mathrm{d}x_2}{\mathrm{d}t}(t; t_0, z) + M^{-1/2}\Pi\big(M^{-1/2}x_1(t; t_0, z)\big), v_2 - x_2(t; t_0, z)\right\rangle_m \\ &\quad + \Phi\big(H_1^{\mathrm{T}} M^{-1/2} v_2\big) - \Phi\big(H_1^{\mathrm{T}} M^{-1/2} x_2(t; t_0, z)\big) \geqslant 0, \quad \forall v_2 \in \mathbb{R}^m,\ \text{a.e. } t \geqslant t_0. \end{aligned}$$

However, $\gamma(z) \subset \mathcal{D} \subset E_\Psi(F, \varphi, V)$ and thus $x_2(t; t_0, z) = 0, \forall t \geqslant t_0$. We deduce that $x_1(\cdot; t_0, z) = z_1,\ \forall t \geqslant t_0$ and

$$\left\langle M^{-1/2}\Pi'\left(M^{-1/2}z_1\right), v_2\right\rangle_m + \Phi\left(H_1^{\mathrm{T}} M^{-1/2} v_2\right) - \Phi(0) \geqslant 0, \quad \forall v_2 \in \mathbb{R}^m,\ \text{a.e. } t \geqslant t_0.$$

Thus $\Pi'(M^{-1/2}z_1) \in -H_1 \partial\Phi(0)$. Assumption (1) yields $\Pi'(M^{-1/2}z_1) = 0$. Recalling that $\|z_1\| \leqslant \sigma$ since $z \in B_\sigma$, we obtain $M^{-1/2}z_1 \in B_{\tilde{\sigma}}$. Assumption (5) gives $z_1 = 0$. It results that $\mathcal{D} = \{0\}$. Thus $\mathcal{M} = \{0\}$ and the attractivity of the trivial stationary solution follows. $\square$

6. Examples in unilateral mechanics

Example 1. The model of Fig. 1 consists of a mass $m > 0$ restrained by a spring with stiffness constant $k > 0$ and a damper with viscous damping coefficient $c > 0$. The motion of the mass has frictional contact. A Coulomb model is assumed for the friction force f, i.e.,

$$f \in -\partial\Phi(\dot{q}),$$

with

$$\Phi(x) = \gamma|x|,$$

where $\gamma > 0$ denotes the coefficient of friction.

The motion of the system is described by the model:

$$m\ddot{u}(t) + c\dot{u}(t) + ku(t) \in -\partial\Phi\left(\dot{u}(t)\right). \tag{64}$$

Here $q = (u)$, $M = (m)$, $K = (k)$, $C = (c)$, $H_1 = (1)$, $H_2 = (1)$, $D(\partial\Phi) = \mathbb{R}$ and $\partial\Phi(0) = [-\gamma, +\gamma]$. Setting $R = (\sqrt{m})$, we see that both assumptions of Theorem 6 hold. The set of stationary solutions is here given by

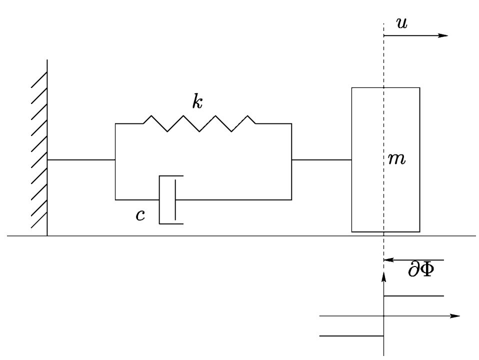

Fig. 1. Example 1.

$$\mathcal{W} = \left[-\frac{\gamma}{k}, +\frac{\gamma}{k}\right].$$

This set defines a steady zone due to friction.

It is also easy to check that both assumptions of Theorems 7 and 9 are satisfied. It results that *each stationary solution* $\bar{q} \in \mathcal{W}$ *is stable*. Moreover,

$$\lim_{\tau \to +\infty} d\big(u(\tau; t_0, q_0, \dot{q}_0), \mathcal{W}\big) = 0$$

and

$$\lim_{\tau \to +\infty} \dot{u}(\tau; t_0, q_0, \dot{q}_0) = 0.$$

Some numerical results ($m = 1, k = 1, c = 0.2, \gamma = 1$) are given in Figs. 2, 3 and 4 so as to illustrate and support this last theoretical result.

Example 2. We consider the model given in Fig. 5. A mass $m > 0$ is restrained by a vertical spring with stiffness constant $k_V > 0$ in parallel with a damper with coefficient of viscous damping $c_V > 0$ and some inclined device formed by a spring with stiffness constant $k_I > 0$ in parallel with a nonlinear damper whose characteristic (feedback force-speed) is described by a monotone set-valued graph $\partial\Phi$ as the one depicted in Fig. 6.

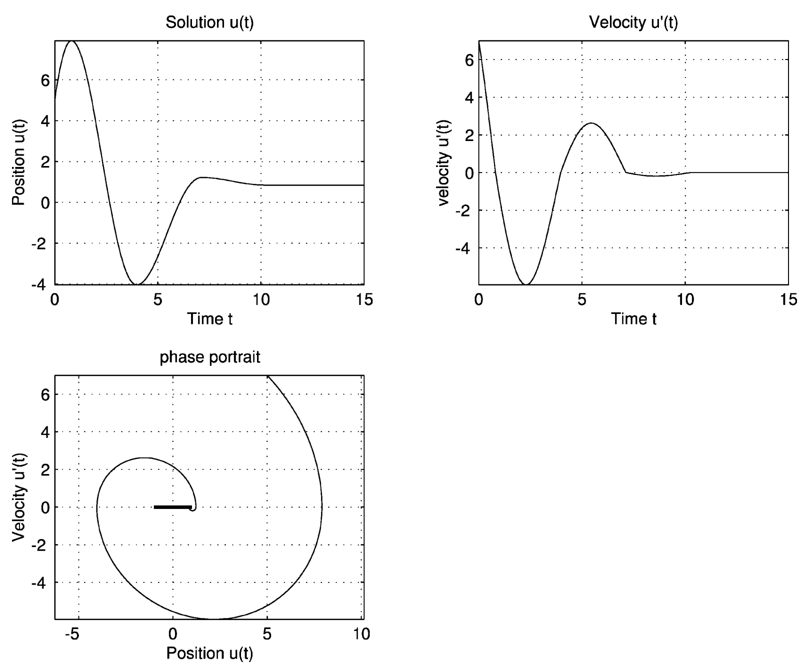

Fig. 2. Example 1.

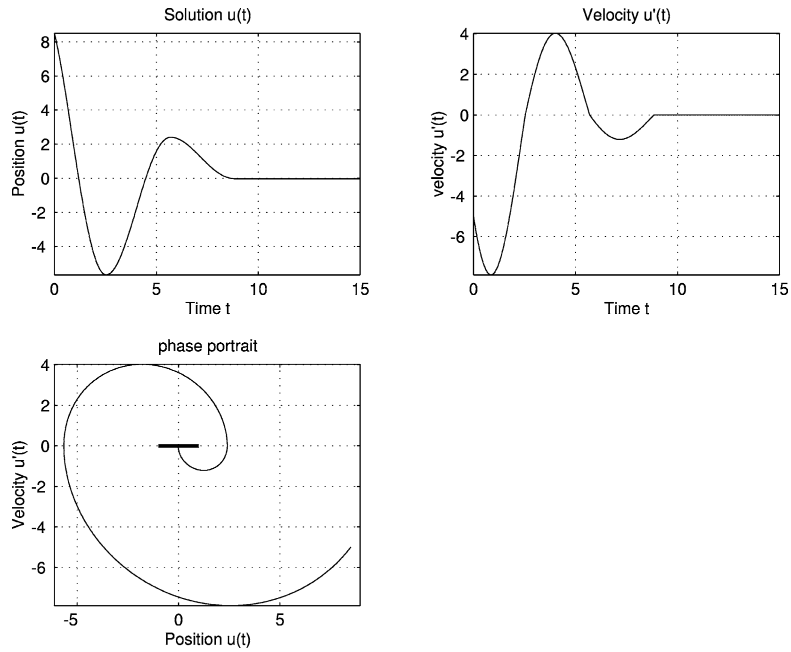

Fig. 3. Example 1.

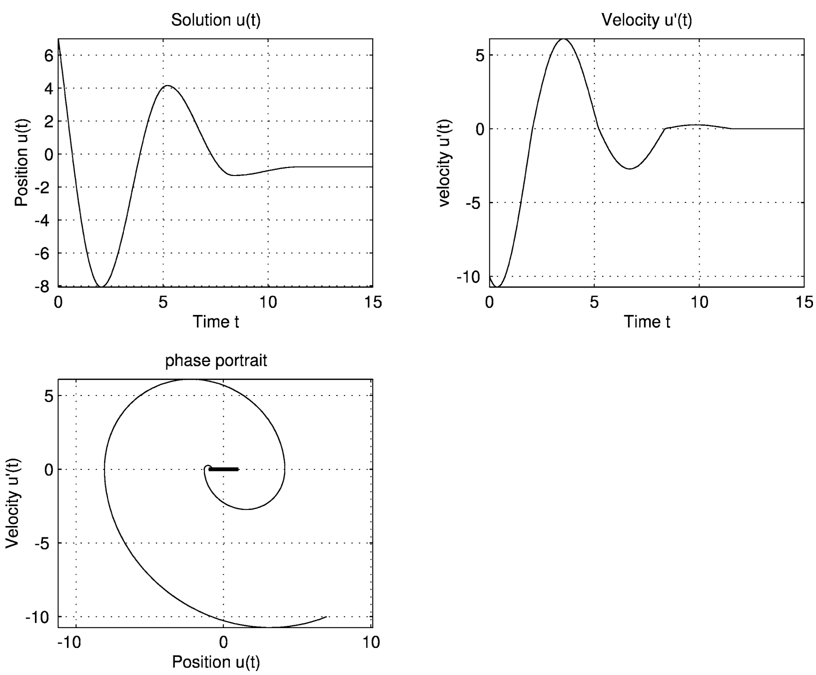

Fig. 4. Example 1.

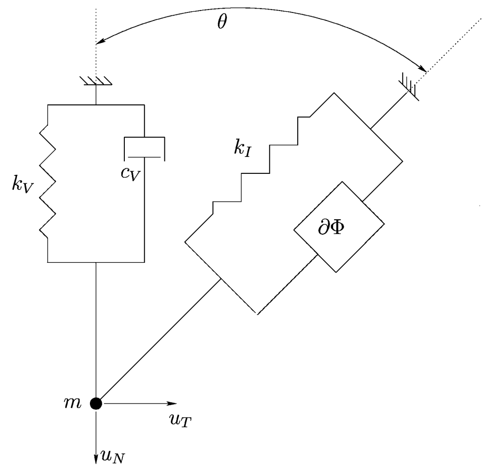

Fig. 5. Example 2.

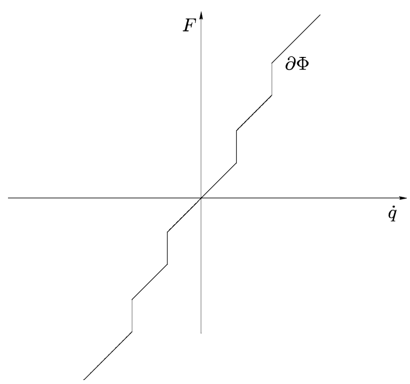

Fig. 6. Multivalued monotone mapping $\partial\Phi$.

The angle of inclination is denoted by $\theta \in (0, \pi/2)$. The horizontal and vertical displacement of the mass m are respectively denoted by u_N and u_T.

The model describing the motion of this system is of the form given in (30) with

$$M = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}, \qquad K = \begin{pmatrix} k_I \sin^2\theta & -k_I \sin\theta\cos\theta \\ -k_I \sin\theta\cos\theta & k_V + k_I \cos^2\theta \end{pmatrix}, \qquad C = \begin{pmatrix} 0 & 0 \\ 0 & c_V \end{pmatrix},$$

$$H_1 = \begin{pmatrix} \sin\theta \\ -\cos\theta \end{pmatrix}, \qquad H_2 = H_1^{\mathrm{T}}, \qquad q = \begin{pmatrix} u_T \\ u_N \end{pmatrix}$$

and with $\Phi : \mathbb{R} \to \mathbb{R}$ as depicted in Fig. 6. Here $D(\partial\Phi) = \mathbb{R}$, $\partial\Phi(0) = \{0\}$ and $\Phi(x) > 0$, $\forall x \neq 0$.

It is clear that all the assumptions of Theorem 6 hold with

$$R = \begin{pmatrix} \sqrt{m} & 0 \\ 0 & \sqrt{m} \end{pmatrix}.$$

The set of stationary solutions reduces here to $\{0\}$ since K is nonsingular and $\partial\Phi(0) = \{0\}$. Thus the trivial stationary solution is here the unique stationary solution. We see that $RM^{-1}CR^{-1}$ is positive semidefinite and $RM^{-1}KR^{-1}$ is symmetric and positive definite.

We may apply Theorem 7 and conclude that the trivial stationary solution is stable.

Let us now check that Theorem 9 can also be applied. It remains to verify that assumption (2) in Theorem 9 holds. We have:

$$\begin{aligned}
&\langle RM^{-1}CR^{-1}z, z\rangle_2 + \Phi\big(H_2R^{-1}z\big) - \Phi(0) \\
&\quad = \frac{c_V}{m}|z_2|^2 + \Phi\left(\frac{1}{\sqrt{m}}\sin(\theta)z_1 - \frac{1}{\sqrt{m}}\cos(\theta)z_2\right).
\end{aligned}$$

It is thus clear that

$$\langle RM^{-1}CR^{-1}z, z\rangle_2 + \Phi\big(H_2R^{-1}z\big) - \Phi(0) \geqslant 0.$$

Suppose now that

$$\langle RM^{-1}CR^{-1}z, z\rangle_2 + \Phi\big(H_2R^{-1}z\big) - \Phi(0) = 0.$$

Then $|z_2|^2 = 0$ and $\Phi(m^{-1/2}\sin(\theta)z_1 - m^{-1/2}\cos(\theta)z_2) = 0$. This yields $z_2 = 0$ and next $z_1 = 0$. Assumption (2) of Theorem 9 is thus satisfied.

Theorem 9 ensures that the trivial stationary solution is globally attractive.

In conclusion, the trivial stationary solution is (a) *the unique stationary solution*, (b) *stable* and (c) *globally attractive*. Properties (b) and (c) entail that the trivial stationary solution is *asymptotically stable*.

A numerical simulation is given in Fig. 7.

Example 3. Let us consider the system of Fig. 8. Here $m > 0$ denotes the mass of a mass point, $l > 0$ is the length of the rod and $k > 0$ is the stiffness of the spiral spring. The angle θ determines the position of the system. The friction force f at the horizontal cylindrical support is given by the model $f \in -\partial\Phi(\dot{\theta})$ where $\Phi : \mathbb{R} \to \mathbb{R}$ is a convex and lower semicontinuous function. The motion of the system is governed by the model:

$$ml^2\ddot{\theta} + k\theta - mgl\sin(\theta) \in -\partial\Phi(\dot{\theta}). \tag{65}$$

Here $q = (\theta)$, $M = (ml^2)$, $H_1 = (1)$ and

$$\Pi(\theta) = \frac{1}{2}k\theta^2 - mgl\big(1 - \cos(\theta)\big).$$

We see that $\Pi(0) = \Pi'(0) = 0$. Moreover if

$$k > mgl, \tag{66}$$

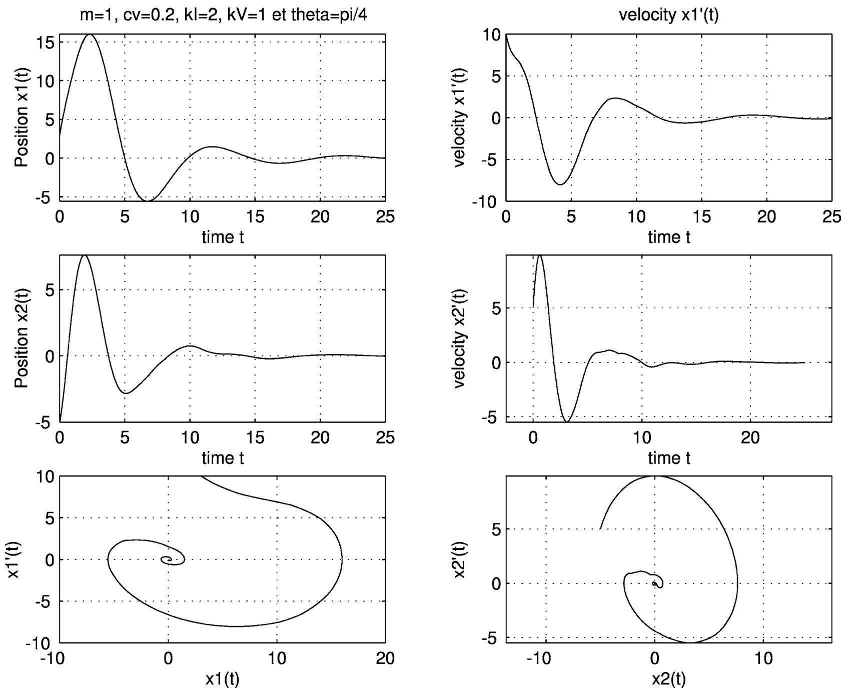

Fig. 7. Example 2.

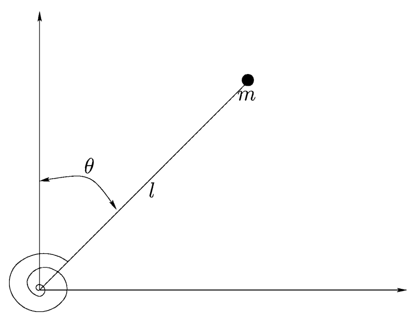

Fig. 8. Example 3.

then there exists $\sigma_1 > 0$ such that $\Pi(\theta) > 0$, $|\theta| \leqslant \sigma_1, \theta \neq 0$. It is also clear that there exists $\sigma_2 > 0$ such that $\Pi'(\theta) \neq 0$, $|\theta| \leqslant \sigma_2, \theta \neq 0$.

Suppose that Φ is of the form given in Example 1. Then $0 \in \partial\Phi(0)$ and Theorem 11 can be applied to ensure that the trivial stationary solution of (65) is *stable*. A numerical simulation is given in Fig. 9.

If Φ is of the form depicted in Example 2, then $\partial\Phi(0) = \{0\}$ and $\Phi(x) > 0, x \neq 0$. It is clear that all the assumptions of Theorem 12 are satisfied. The trivial stationary solution of (65) is thus in this case *asymptotically stable*.

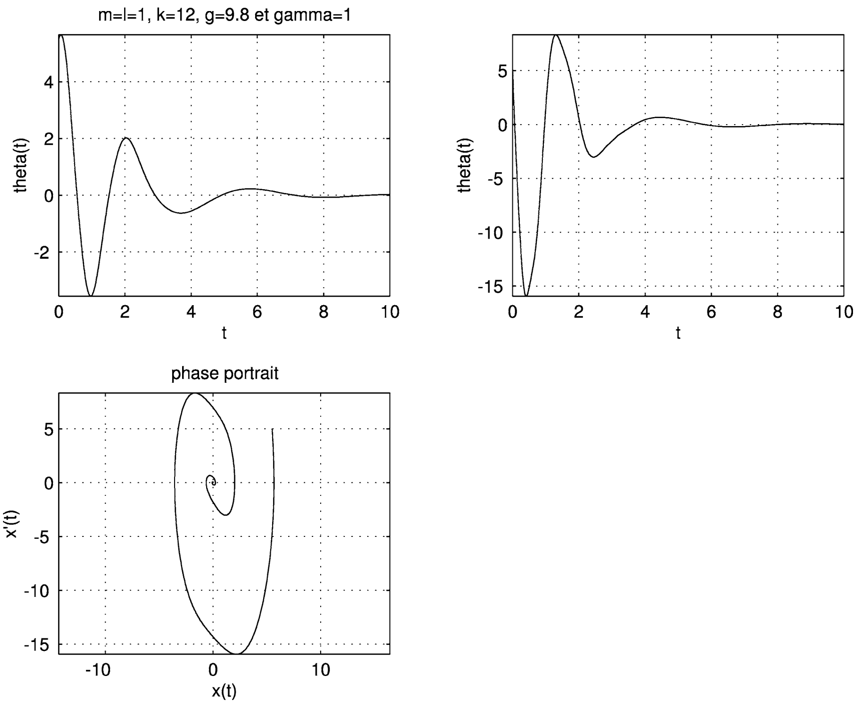

Fig. 9. Example 3.

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